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1979 J. Phys. A: Math. Gen. 12 21

(http://iopscience.iop.org/0305-4470/12/1/013)

J. Phys. A: Math. Gen., Vol. 12, No. 1, 1979. Printed in Great Britain

Computing the invariants of Lie superalgebras

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Received 8 May 1978

Abstract. Lie superalgebras (LS) generalise the class of Lie algebras (LA) by admitting a bracket multiplication which is sometimes commutative and sometimes anticommutative. It is known that such algebras can be represented by first-order Grassmann differential operators acting on Euclidean superfields. This leads, by generalising the usual procedure for LA, to a method for computing the Casimir invariants of any LS: first solve a system of Grassman differential equations to obtain the invariants of the graded symmetric algebra, and then map onto the centre of the enveloping algebra by means of the graded symmetrising isomorphism.

1. Introduction

The non-group theoretical boson–fermion equivalences in the supersymmetric theories of elementary particle physics continue to exercise theoretical physicists. One's intuitive unease with such models, however, has been at least partially overcome by the remarkable successes of the associated ideas and formalism in the theories of supergravity, for example see Freedman *et al* (1976), Deser and Zumino (1976). These achievements help to sustain interest and spur development in the mathematical physics literature for the now well established field of Lie superalgebras (also called z_2-graded Lie algebras)—the mathematical abstraction of the underlying symmetry algebras of supersymmetry.

We recall from Corwin *et al* (1975) that a Lie superalgebra (LS) L is a direct sum $L_0 \oplus L_1$ of real or complex vector spaces L_0, the even subspace, and L_1, the odd subspace. We attach a sign function σ to the set $H = L_0 \cup L_1$ of homogeneous elements, defined by $\sigma(l) = 0$ if $l \in L_0$, but 1 if $l \in L_1$. L possesses a multiplication, denoted by [,], which satisfies

$$[L_0, L_0] \subseteq L_0; \qquad [L_0, L_1] = [L_1, L_0] \subseteq L_1; \qquad [L_1, L_1] \subseteq L_0; \quad \text{(A1)}$$

$$[l, m] = -(-1)^{\sigma(l)\sigma(m)}[m, l] \qquad \text{for all } l, m \in H; \quad \text{(A2)}$$

$$[l, [m, n]] = [[l, m], n] + (-1)^{\sigma(l)\sigma(m)}[m, [l, n]] \qquad \text{for all } l, m, n \in H. \quad \text{(A3)}$$

The LS which have occurred in supersymmetric theories and also some which occur in the classification scheme for simple LS of Kac (1977) are expressly given as algebras of linear differential operators, with both ordinary (real) and Grassman (anticommuting) variables, acting on Grassman algebra-valued functions (superfields), and which in supersymmetry leave invariant a super-Lagrangian. There is therefore some interest in exploring and determining the sort of invariants that arbitrary LS can possess. In a previous paper, Backhouse (1977a), as part of a study of Killing forms, we showed how to compute series of homogeneous Casimir invariants for certain LS. These invariants lie in the centre of the enveloping algebra—the associative algebra generated by the LS and within which the bracket multiplication has interpretation, for homogeneous generators, as a commutator or an anticommutator. Unfortunately the procedure was cumbersome even for the simplest of examples, and could only be used for the small number of LS which possess a non-degenerate second-order invariant form. It is the aim of the present paper to establish and illustrate a method which can be applied to any LS to yield all of its Casimir invariants. This provides a generalisation of the well known theory for Lie algebras (LA), whose Casimir invariants can be found by solving systems of first-order linear partial differential equations. An account of the latter method can be found in a recent article by Patera *et al* (1976), wherein its facility has been well illustrated. As shown by these authors, the procedure for LA does more than one originally asked of it, for it can produce invariants more exotic than those of Casimir (polynomial) type, which can lie even outside of the quotient field of the enveloping algebra. Of course a generalisation of this experience is a feature of our work here on LS.

Our procedure falls into two parts, the first analytic and the second algebraic in nature. We stated above that many LS have been given explicitly as algebras of Grassmann differential operators—we call them superdifferential operators. It is in fact true that any LS can be so represented. In § 2 we make this more precise and remind ourselves of the algebraic properties of the operators of the appropriate Grassmann or superdifferential calculus. The analytic part of our procedure for computing the invariants of a given LS, L say, is to seek superfields which are annihilated by the associated algebra of superdifferential operators. Some of these solutions, those whose coefficients are polynomials in the real variables, are the invariants of the graded symmetric algebra $S(L)=S(L_0)\otimes\Lambda(L_1)$, where $S(L_0)$ is the symmetric algebra of L_0 and $\Lambda(L_1)$ is the alternating or Grassmann algebra generated by L_1. In § 3 we produce a symmetrising map which takes $S(L)$ isomorphically onto $U(L)$, the enveloping algebra of L, and has the property of taking invariants into invariants. We illustrate this method in § 4, using as examples the ubiquitous di-spin algebra and some of the low-dimensional LS recently classified by us, Backhouse (1978).

2. Superdifferential equations

We recall that the Grassmann algebra G_q, of dimension 2^q, is generated by an identity and a set of q independent anticommuting elements $\sigma_1, \sigma_2, \ldots, \sigma_q$. G_q has basis $\{1$ and all $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_r}$: $1\leq i_1<i_2<\ldots<i_r\leq q$, $1\leq r\leq q\}$. G_q is determined by the relations $\sigma_i\sigma_j+\sigma_j\sigma_i=0$ for all i, j, a particular case of which is $\sigma_i^2=0$. If we wish to emphasise that G_q is the alternating algebra of a particular vector space V of dimension q, then we write $G_q=\Lambda(V)$. The latter is formally constructed as the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by the second rank tensors of the form $x\otimes y+y\otimes x$, for all $x, y\in V$. The generating basis $\{\sigma_i\}$ is the image of a basis for V, included in $T(V)$, under the canonical map $T(V)\rightarrow\Lambda(V)$.

In the literature of supersymmetry we find expressions of the form

$$F=f_0 1+\sum_i f_i\sigma_i+\sum_{i<j} f_{ij}\sigma_i\sigma_j+\ldots f_{12\ldots q}\sigma_1\sigma_2\ldots\sigma_q, \qquad (1)$$

where the f are real or complex functions of real variables $x=(x_1, x_2, \ldots, x_p)$, say. Such expressions have different interpretations according to taste and context. On the one hand (1) defines a G_q-valued function on R^p, which we refer to as a Euclidean superfield. On the other hand the space of all expressions (1) forms an associative algebra under the obvious addition and multiplication. We shall presently define certain linear operators which act on a subalgebra of this algebra, but which in the superfield interpretation are differentiations. It is immaterial in the context of the present paper whether we think we are doing linear algebra on an associative algebra or whether we think we are doing calculus on Euclidean superfields. It is important to note, however, that it is possible to define superfields over manifolds and then considerable care must be taken over the handling of expressions of the form (1)—see the article of Kostant (1977) on graded manifolds and graded Lie groups.

If the coefficient functions f of the superfield F are sufficiently differentiable we can define new superfields $\partial F/\partial x_i$, $\partial^2 F/\partial x_i\, \partial x_j$, etc, simply by differentiating the coefficients. More interestingly, however, is the possibility of differentiating with repect to the anticommuting 'variables'. We define differential operators $\partial/\partial\sigma_i$, $\partial^2/\partial\sigma_i\,\partial\sigma_j$, etc, as follows. If F is of the form $x\to f_0(x)1$, then $\partial F/\partial\sigma_i$ is the zero superfield. If F is of the form $x\to f_j(x)\sigma_j$, then $\partial F/\partial\sigma_i$ is the superfield $x\to f_j(x)\delta_{ij}1$. If F is of the form $x\to f_{jk}(x)\sigma_j\sigma_k$, then $\partial F/\partial\sigma_i$ is the superfield $x\to f_{jk}(x)(\delta_{ij}\sigma_k-\delta_{ik}\sigma_j)$. More algebraically we can write

$$\frac{\partial}{\partial\sigma_i}(f(x))=0, \qquad \frac{\partial}{\partial\sigma_i}\sigma_j=\delta_{ij}, \qquad \frac{\partial}{\partial\sigma_i}(\sigma_j\sigma_k)=\delta_{ij}\sigma_k-\delta_{ik}\sigma_j.$$

Generally, we have an alternating rule of signs for differentiating products of σ's. The higher-order superdifferential operators satisfy mixed derivative restrictions of which the simplest are $\partial^2/\partial\sigma_i\,\partial\sigma_j=-\partial^2/\partial\sigma_j\,\partial\sigma_i$ and in particular $\partial^2/\partial\sigma_i^2=0$. Superdifferentiations commute with ordinary differentiations and we have higher-order derivatives of the form $\partial^2/\partial x_i\,\partial\sigma_j$, etc. General linear superdifferential operators are constructed by taking linear combinations of these basic ones with superfield coefficients. The notion of a superdifferential equation $\mathrm{D}F=F'$, where D is a superdifferential operator and F, F' are superfields, follows immediately.

Now let $L=L_0\oplus L_1$ be a LS, where the even space L_0 has basis $a_1, a_2, \ldots, a_p$, and the odd space L_1 has basis $\alpha_1, \alpha_2, \ldots, \alpha_q$. The graded Lie bracket is determined by the commutation/anticommutation relations

$$[a_i, a_j]=\sum_{k=1}^{p} C_{ik}^{k}a_k, \tag{2}$$

$$[a_i, \alpha_j]=-[\alpha_j, a_i]=\sum_{k=1}^{q} D_{ij}^{k}\alpha_k, \tag{3}$$

$$[\alpha_i, \alpha_j]=\sum_{k=1}^{p} E_{ij}^{k}a_k, \tag{4}$$

where the structure constants satisfy $C_{ij}^k=-C_{ji}^k$, $E_{ij}^k=E_{ji}^k$, for all relevant i, j, k, and also some more complicated relations obtained from the graded Jacobi conditions (A3).

With the same notation as before construct the superdifferential operators

$$P(a_i)=\sum_{j,k=1}^{p} C_{ij}^{k}x_k\frac{\partial}{\partial x_j}+\sum_{j,k=1}^{q} D_{ij}^{k}\sigma_k\frac{\partial}{\partial\sigma_j}, \qquad \text{for } i=1, 2, \ldots, p; \tag{5}$$

$$Q(\alpha_j) = -\sum_{i,k=1}^{p,q} D_{ij}^{k}\sigma_k \frac{\partial}{\partial x_i} + \sum_{i,k=1}^{q,p} E_{ji}^{k} x_k \frac{\partial}{\partial \sigma_i}, \qquad \text{for } j = 1, 2, \ldots, q. \quad (6)$$

Then using the graded Jacobi relations it is easy to check that the operators $\{P(a_i)\}$ and $\{Q(\alpha_j)\}$ satisfy the same commutation/anticommutation relations (2), (3), (4) as do the generators $\{a_i\}$, $\{\alpha_j\}$ of L. Furthermore, these operators actually provide us with the adjoint action of L on its graded symmetric algebra $S(L) = S(L_0) \otimes \Lambda(L_1)$, considered as a subalgebra of the algebra of superfields. To understand and prove this we recall first of all that the graded symmetric algebra $S(L)$ of L is formally constructed as the quotient of the tensor algebra $T(L)$ by the two-sided ideal generated by second rank tensors $l \otimes m - (-1)^{\sigma(l)\sigma(m)} m \otimes l$ for all $l, m \in L$. Then the symmetric algebra can be considered as superfields whose coefficient functions are polynomials. The adjoint action of L on itself extends to $T(L)$, preserves the ideal, and hence passes to the quotient, uniquely as a graded derivation. The operators (5), (6) are also graded derivations of the symmetric algebra, so it suffices, in order to prove that their action coincides with the adjoint action, to check that it is correct at the generator level. This is immediate from the calculations

$$P(a_i)x_j = \sum_{k=1}^{p} C_{ij}^{k} x_k, \qquad P(a_i)\sigma_j = \sum_{k=1}^{q} D_{ij}^{k} \sigma_k,$$

$$Q(\alpha_i)x_j = -\sum_{k=1}^{q} D_{ji}^{k} \sigma_k, \qquad Q(\alpha_i)\sigma_j = \sum_{k=1}^{p} E_{ij}^{k} x_k.$$

It should be noted in these equations that there is a one-to-one correspondence between the generators x_i, σ_j of $S(L)$ and the basis elements a_i, α_j of L.

We say that an element of the symmetric algebra is an invariant if it is annihilated by all elements of L under the adjoint action. It follows from what we have said above that $u \in S(L)$ is an invariant if and only if $P(a_i)u = Q(\alpha_j)u = 0$ for $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. The search for invariants therefore devolves on the solution of systems of superdifferential equations. It should be noted, however, that such equations often have many solutions whose coefficient functions are not polynomials and hence lie outside of the symmetric algebra.

3. Casimir invariants

The second step in our programme for calculating Casimir invariants is the mapping of the centre of the graded symmetric algebra onto the centre of the enveloping algebra. The enveloping algebra $U(L)$ is formally constructed as the quotient of the tensor algebra $T(L)$ by the two-sided ideal generated by tensors of the form $l \otimes m - (-1)^{\sigma(l)\sigma(m)} m \otimes l - [l, m]$, for all homogeneous $l, m \in L$. $U(L)$ contains L and is such that the graded Lie bracket on L is represented by a mixture of commutators and anticommutations, as appropriate. By extension of the adjoint action of L on itself, L acts as a set of graded derivations on $U(L)$, and then the Casimir invariants are those elements of $U(L)$ which are annihilated by all of L.

It was first proved by Ross (1965) that $U(L)$ has as basis the set of elements $\{a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_p^{\lambda_p} \alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \ldots \alpha_q^{\epsilon_q} : \lambda_i$ non-negative integers and $\epsilon_i = 0$ or $1\}$. It is also easily seen that the graded symmetric algebra $S(L)$ has basis the set of elements $\{x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_p^{\lambda_p} \sigma_1^{\epsilon_1} \sigma_2^{\epsilon_2} \ldots \sigma_q^{\epsilon_q} : \lambda_i$ non-negative integers and $\epsilon_i = 0$ or $1\}$. $U(L)$ and $S(L)$

are manifestly isomorphic as vector spaces, though in general, not as algebras. Unfortunately this isomorphism does not carry invariants of $S(L)$ into invariants of $U(L)$. The root of this failure is the lack of symmetry in $U(L)$ with respect to reordering the basis of L. This can be overcome by use of the graded symmetrising map which we now describe, first by example and then in general. Because $x_1x_2 = x_2x_1$, but $a_1a_2 \neq a_2a_1$ in general, we cannot have $x_1x_2 \to a_1a_2$ and $x_2x_1 \to a_2a_1$, so we compromise by mapping x_1x_2, hence x_2x_1, to $\frac{1}{2}(a_1a_2 + a_2a_1)$. Similarly we map $\sigma_1\sigma_2$ to $\frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1)$. More generally let $u = y_1 \otimes y_2 \otimes \ldots \otimes y_n$ be a homogeneous element of $T(L)$ of degree n, where r of the y's belong to L_0 and s $(= n - r)$ of the y's belong to L_1. The odd elements will be distributed in some particular order in u. Now let π be an element of the symmetric group S_n, then define $\pi u = \chi_\pi(u) y_{\pi^{-1}(1)} y_{\pi^{-1}(2)} \ldots y_{\pi^{-1}(n)}$, where $\chi_\pi(u)$ is ± 1 determined as follows: π rearranges the y's and in particular rearranges the odd elements. Then $\chi_\pi(u)$ is the parity of the permutation of degree s which would restore the odd elements to their original order relative to themselves. For example if $u = \alpha_1 \otimes a_1 \otimes \alpha_2$ and $\pi = (1\ 2)(3)$, then $\pi u = a_1\alpha_1\alpha_2$, but if $\pi = (1\ 2\ 3)$, then $\pi u = -\alpha_2\alpha_1 a_1$. If $u \in T(L)$ is a monomial of degree n we can map it to $U(L)$ and totally symmetrise it by forming

$$Su = \frac{1}{n!} \sum_{\pi \in S_n} \pi u. \tag{7}$$

If u is not necessarily a monomial then it can be uniquely written as a sum of monomials of possibly different degrees, and we can separately symmetrize each component. The mapping $S: T(L) \to U(L)$ so defined clearly induces a unique map $S(L) \to U(L)$, which we shall denote by the same symbol S. The manifest isomorphism between $S(L)$ and $U(L)$ which we observed above can be represented by the identity matrix. The transformation S, just defined, has a triangular matrix with unity down the diagonal, and therefore provides an isomorphism. We claim that if U is an invariant of $S(L)$, then SU is an invariant of $U(L)$. This follows from the following theorem.

Theorem. Let D be a graded derivation of L. Denote also by D its unique extensions, as graded derivations, to $S(L)$ and $U(L)$. Then $SD = DS$.
Proof. It suffices to show that $SD - DS$ annihilates monomials.

Let u be a monomial of degree n. It is clear that each of SDu and DSu can be developed as a set of $n \,.\, n!$ monomials, and, furthermore, that the two sets agree except possibly in the signs attached to the terms. The problem is to check that the signs do in fact agree perfectly.

Suppose $u \in S(L)$ can be represented by $y_1 \otimes y_2 \otimes \ldots \otimes y_n \in T(L)$, using our previous notation. The unsigned terms of SDu and DSu are of the form $y_{\pi^{-1}(1)} y_{\pi^{-1}(2)} \ldots y_{\pi^{-1}(i-1)} D y_{\pi^{-1}(i)} y_{\pi^{-1}(i+1)} \ldots y_{\pi^{-1}(n)}$. This can be regarded as a concatenation of the result of applying a permutation of degree $n+1$ to the ordered graded objects $\{D, y_1, y_2, \ldots, y_n\}$. D acts on the y which appears immediately to its right and then all of the factors are multiplied together. The permutation which achieves this appears factorised in two different ways according as the monomial is obtained using SD or DS. The point is that the sign which should be attached to the monomial is the parity of the permutation which rearranges the odd elements of the ordered set $\{D, y_1, y_2, \ldots, y_n\}$. The parity is independent of just how the permutation is factorised.

Corollary. Let u be an element of $S(L)$. Then u is an invariant in $S(L)$ if and only if Su is an invariant in $U(L)$.

Proof. If $l \in H$, then D_l, defined by $D_l m = [l, m]$, for all $m \in L$, is a graded derivation of L.

If $u \in S(L)$ is an invariant then $D_l Su = SD_l u = 0$, for all $l \in H$, by the theorem and the invariance of u. Therefore Su is annihilated by all homogeneous elements of L, so it is annihilated by all elements of L, which means that Su is an invariant in $U(L)$. The converse is also true.

Before discussing specific examples we note two features of $S(L)$ which we sometimes use to simplify even further the search for its invariants and those of $U(L)$. The first concerns its double grading. In the first place $S(L)$ is graded by the non-negative integers: if we put $S^{(n)}(L)$ equal to

$$\text{span}\left\{x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_p^{\lambda_p}\sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}\ldots\sigma_q^{\epsilon_q}\colon \sum_{i=1}^{p}\lambda_i+\sum_{j=1}^{q}\epsilon_j=n\right\},$$

then the span of $\{u_1u_2\colon u_1 \in S^{(m)}(L), u_2 \in S^{(n)}(L)\}$ is $S^{(m+n)}(L)$. It is clear that $S(L) \underset{n\geq 0}{=} s^{(n)}(L)$ and that each space of homogeneous elements $S^{(n)}(L)$ is stable under derivation. It follows that each invariant of $S(L)$ can be uniquely written as a linear combination of homogeneous invariants.

Further to this grading we have a finer two-fold grading. Let ϕ be the grading automorphism of L, defined by $\phi(l_0+l_1)=l_0-l_1$, where $l_0 \in L_0$, $l_1 \in L_1$. ϕ satisfies $\phi^2=1$ and $\phi[l, m]=[\phi(l), \phi(m)]$ for all $l, m \in L$. ϕ extends to $T(L)$ and passes to the quotient $S(L)$ as an involutive automorphism. The eigenspaces $S(L)_0$ and $S(L)_1$ which decompose $S(L)$ are easily defined by their bases. A basis for $S(L)_{0(1)}$ has $\Sigma_{j=1}^{q}\epsilon_j=$ even (odd). Now $S(L)_0$ and $S(L)_1$ are both stable under even derivations but are mapped into one another by odd derivations. It follows that an invariant $u \in S(L)$ can be uniquely written $u=u_0+u_1$, where $u_0 \in S(L)_0$, $u_1 \in S(L)_1$ are both invariants, which we call the even and odd components of u, respectively. It follows from this and what we said above that it suffices to seek the even and odd homogeneous invariants in $S(L)$ to obtain a complete picture of the Casimir invariants of L in $U(L)$. The notion of evenness and oddness of Casimir invariants also makes sense, see Backhouse (1977b), but, in general, the degree of homogeneity is not defined.

Our final remarks concern the use of generating sets for invariants. Suppose there exists a finite set $\{F_1, F_2, \ldots, F_n\}$ of invariants in $S(L)$, which are functionally independent and which generate the algebra of invariants. By generate, we mean that any invariant can be written as a polynomial in the F_i. We claim that $\{SF_1, SF_2, \ldots, SF_n\}$ generate the algebra of Casimir invariants. To prove this let G be a Casimir invariant in $U(L)$. Then, because S is an isomorphism, $S^{-1}G$ is an invariant in $S(L)$. By hypothesis we can write $S^{-1}G=\Lambda(F_1, F_2, \ldots, F_n)$ for some polynomial Λ. Consider $G'=\Lambda(SF_1, SF_2, \ldots, SF_n)$. Clearly this lies in $U(L)$, and, being a function of Casimir invariants, is itself a Casimir invariant. In general G does not equal G' but they do agree in their highest order terms. To see this we only have to recall that, with respect to standard bases, the symmetrising map S has a triangular matrix with ones on the diagonal. So S fails to be an algebra isomorphism only by the introduction of lower-order terms. It follows that $G-G'$ is a Casimir invariant of degree strictly less than the degree of G. Induction completes the argument—the induction is anchored by the observation that S maps a constant superfield to a multiple of the identity in $U(L)$.

4. Examples

In a forthcoming paper, Backhouse (1978), we have classified all real LS of dimension up to four. We now compute the Casimir invariants of a selection of these algebras and also for the di-spin algebra. As far as notation is concerned, elements of L_0 (or L_1) will be denoted by Latin (or Greek) letters taken from the beginning of the alphabet. The letters x, y, z (or σ, τ) will denote real (or anticommuting) variables. Other letters will denote real parameters.

4.1. *Example 1*

The two-dimensional algebra B has basis $\{a, \alpha\}$ and is defined by the relation $[a, \alpha] = \alpha$. The superdifferential operator representation of the algebra is

$$P(a) = \sigma\frac{\partial}{\partial\sigma}, \qquad Q(\alpha) = -\sigma\frac{\partial}{\partial x}. \tag{8}$$

Clearly the only even superfield $x \to f(x)$ which is annihilated by the operators (8) has $f(x) = \text{constant}$. There is no odd superfield $x \to f(x)\sigma$ which is annihilated by (8). It follows the B only has scalar multiples of the identity as invariants.

4.2. *Example 2*

The three-dimensional algebra C_p^1 has basis $\{a, b, \alpha\}$ and is defined by the relations $[a, b] = b$, $[a, \alpha] = p\alpha$, $p \neq 0$. Its operator representation is

$$P(a) = y\frac{\partial}{\partial y} + p\sigma\frac{\partial}{\partial\sigma}, \qquad P(b) = -y\frac{\partial}{\partial x}, \qquad Q(\alpha) = -p\sigma\frac{\partial}{\partial x}. \tag{9}$$

The constant superfields are the only even invariants. An odd invariant $(x, y) \to f(x, y)\sigma$ satisfies $\partial f/\partial x = 0$ and $y(\partial f/\partial y) + pf = 0$. It follows that $f(x, y)$ is proportional to y^{-p}. So, if p is not a negative integer, there are no invariants of C_p^1, other than scalar multiples of the identity. If, however, $p = -n$, where n is a positive integer, then C_{-n}^1 has an odd invariant $y^n\sigma$ in its symmetric algebra. Applying the symmetrising map to $y^n\sigma$, noting that b commutes with α, we obtain the Casimir invariant $b^n\alpha$.

4.3. *Example 3*

The four-dimensional algebra $(C_{-1}^2 + A)$, given as the Jordan–Wigner quantisation algebra by Corwin *et al* (1975), has basis $\{a, b, \alpha, \beta\}$ and is defined by the relations $[a, \alpha] = \alpha$, $[a, \beta] = -\beta$, $[\alpha, \beta] = b$. The operator representation is

$$\begin{aligned} P(a) &= \sigma\frac{\partial}{\partial\sigma} - \tau\frac{\partial}{\partial\tau}, \qquad & P(b) &= 0, \\ Q(\alpha) &= -\sigma\frac{\partial}{\partial x} + y\frac{\partial}{\partial\tau}, \qquad & Q(\beta) &= \tau\frac{\partial}{\partial x} + y\frac{\partial}{\partial\sigma}. \end{aligned} \tag{10}$$

Odd superfields are of the form $(x, y) \to f(x, y)\sigma + g(x, y)\tau$. If such a superfield is annihilated by $P(a)$ then $f(x, y)\sigma - g(x, y)\tau = 0$, so $f = g = 0$. Therefore there are no odd invariants.

Even superfields are of the form F: $(x, y) \to f(x, y) + g(x, y)\sigma\tau$. Surprising at first sight is the fact that $P(a)F = 0$ gives no information. However, both $Q(\alpha)F = 0$ and $Q(\beta)F = 0$, give $(\partial f/\partial x) + yg = 0$. If both f and g are to be polynomials, then f has y as a factor and we can write $f = yh(x, y)$, $g = -\partial h/\partial x$. Therefore the invariants in the graded symmetric algebra are of the form

$$yh(x, y) - \frac{\partial h}{\partial x}\sigma\tau, \qquad (11)$$

where h is an arbitrary polynomial in (x, y). It would be rather messy to explicitly map such an invariant to a Casimir invariant by means of the symmetrising map. Fortunately we have that the invariant (11) can be regarded as a function of the simpler invariants y and $xy - \sigma\tau$. Indeed these form a set of independent generators for the algebra of invariants. Furthermore, their images under the symmetrising map, b and $\frac{1}{2}(ab + ba) - \frac{1}{2}(\alpha\beta - \beta\alpha)$, form a set of independent generators for the algebra of Casimir invariants. We see that we can choose two independent generators to be b and $ab - \alpha\beta$.

Corwin *et al* (1975) has the identification a = number operator N, $b = 1$, $\alpha = \alpha^+$, $\beta = \alpha$, where α^+, α are creation and annihilation operators for a fermion. In this representation b takes the constant value 1 and $ab - \alpha\beta$ takes the constant value zero.

4.4. *Example 4*

Our final example, the di-spin algebra, which is at the heart of spin change in supersymmetry, has already been well worked over, see Corwin *et al* (1975), Pais and Rittenberg (1975), Backhouse (1977a, b). However, the computation of its well known single independent Casimir invariant falls so easily to our new approach, that it seems worthwhile repeating the exercise.

L_0 has basis $\{L_1, L_2, L_3\}$, L_1 has basis $\{q_{1/2}, q_{-1/2}\}$, and the relations are

$$\begin{aligned}
&[L_1, L_2] = L_3, \qquad [L_2, L_3] = L_1, \qquad [L_3, L_1] = L_2;\\
&[L_1, q_{1/2}] = \tfrac{1}{2}\mathrm{i}q_{-1/2}, \qquad [L_1, q_{-1/2}] = \tfrac{1}{2}\mathrm{i}q_{1/2};\\
&[L_2, q_{1/2}] = \tfrac{1}{2}q_{-1/2}, \qquad [L_2, q_{-1/2}] = -\tfrac{1}{2}q_{1/2};\\
&[L_3, q_{1/2}] = \tfrac{1}{2}\mathrm{i}q_{1/2}, \qquad [L_3, q_{-1/2}] = -\tfrac{1}{2}\mathrm{i}q_{-1/2};\\
&[q_{1/2}, q_{1/2}] = L_2 + \mathrm{i}L_1,\\
&[q_{-1/2}, q_{-1/2}] = L_2 - \mathrm{i}L_1, \quad [q_{1/2}, q_{-1/2}] = -\mathrm{i}L_3.
\end{aligned} \qquad (12)$$

The operator representation is

$$\begin{aligned}
P(L_1) &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + \frac{\mathrm{i}}{2}\tau\frac{\partial}{\partial\sigma} + \frac{\mathrm{i}}{2}\sigma\frac{\partial}{\partial\tau};\\
P(L_2) &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} + \frac{1}{2}\tau\frac{\partial}{\partial\sigma} - \frac{1}{2}\sigma\frac{\partial}{\partial\tau};\\
P(L_3) &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\mathrm{i}}{2}\sigma\frac{\partial}{\partial\sigma} - \frac{\mathrm{i}}{2}\tau\frac{\partial}{\partial\tau};
\end{aligned} \qquad (13)$$

$$Q(q_{1/2}) = -\frac{i}{2}\tau\frac{\partial}{\partial x} - \frac{1}{2}\tau\frac{\partial}{\partial y} - \frac{1}{2}\sigma\frac{\partial}{\partial z} + (y+ix)\frac{\partial}{\partial\sigma} - iz\frac{\partial}{\partial\tau};$$

$$Q(q_{-1/2}) = -\frac{i}{2}\sigma\frac{\partial}{\partial x} + \frac{1}{2}\sigma\frac{\partial}{\partial y} + \frac{i}{2}\tau\frac{\partial}{\partial z} + (y-ix)\frac{\partial}{\partial\tau} - iz\frac{\partial}{\partial\sigma}.$$

Suppose an odd invariant superfield $(x, y, z) \to f(x, y, z)\sigma + g(x, y, z)\tau$ exists. Applying $Q(q_{1/2})$ to it and equating the result to zero we get

$$-\frac{i}{2}\frac{\partial f}{\partial x}\tau\sigma - \frac{1}{2}\frac{\partial f}{\partial y}\tau\sigma - \frac{i}{2}\frac{\partial g}{\partial z}\sigma\tau + (y+ix)f - izg = 0. \tag{14}$$

This equation must hold for all (x, y, z), so we deduce $f = g = 0$. Therefore there are no odd invariant superfields.

Now let $(x, y, z) \to f(x, y, z) + g(x, y, z)\sigma\tau$ be an even invariant superfield. In particular it is annihilated by $P(L_1)$, $P(L_2)$, $P(L_3)$. A simple calculation shows that

$$J_1 f + J_1 g\sigma\tau = J_2 f + J_2 g\sigma\tau = J_3 f + J_3 g\sigma\tau = 0,$$

where

$$J_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad J_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad J_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

This implies $J_1 f = J_2 f = J_3 f = J_1 g = J_2 g = J_3 g = 0$. We deduce $f = F(x^2 + y^2 + z^2)$, $g = G(x^2 + y^2 + z^2)$. The invariant is also annihilated by $Q(q_{1/2})$, $Q(q_{-1/2})$, which lead to

$$\frac{\partial f}{\partial x} = -2xg, \qquad \frac{\partial f}{\partial y} = -2yg, \qquad \frac{\partial f}{\partial z} = -2zg.$$

Suppose f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf,$$

from which we deduce $(x^2 + y^2 + z^2)g = -\frac{1}{2}nf$. So if $f = (x^2 + y^2 + z^2)^{n/2}$ then $g = -\frac{1}{2}n(x^2 + y^2 + z^2)^{(n/2)-1}$. The most general invariant of degree n is therefore

$$(x^2 + y^2 + z^2)^{n/2} - \tfrac{1}{2}n(x^2 + y^2 + z^2)^{(n/2)-1}\sigma\tau. \tag{15}$$

But this can be written as $[(x^2 + y^2 + z^2) - \sigma\tau]^{n/2}$. The even invariants are generated by $x^2 + y^2 + z^2 - \sigma\tau$. The generator for the Casimir invariants is

$$L_1^2 + L_2^2 + L_3^2 - \tfrac{1}{2}(q_{1/2}q_{-1/2} - q_{-1/2}q_{1/2}), \tag{16}$$

as expected from previous calculations.

Acknowledgments

The author is grateful to the Indian Universities Grants Commission and the British Council for an award under the Younger Scientists Exchange Scheme which enabled him to visit several scientific institutions in India, during winter 1977/8. It is a pleasure to thank the Professors and staff of the Theoretical Physics Department, the University of Madras, of the Tata Institute for Fundamental Research, Bombay, and of Matscience, Madras, for their kind hospitality and encouragement.

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